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# Addendum to 'Coherent Lagrangian vortices: the black holes of turbulence'

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In Haller & Beron-Vera (*J. Fluid Mech.*, vol. 731, 2013, R4) we developed a variational principle for the detection of coherent Lagrangian vortex boundaries. The solutions of this variational principle turn out to be closed null geodesics of the Lorentzian metric induced by a generalized Green–Lagrange strain tensor family. This metric interpretation implies a mathematical analogy between coherent Lagrangian vortex boundaries and photon spheres in general relativity. Here, we give an improved discussion of this analogy.

**Key words:** geophysical and astrophysical flows, geostrophic turbulence, ocean circulation

## 1. The main results of Haller & Beron-Vera (2013)

We consider a two-dimensional velocity field $v(x, t)$, with $x$ labelling the location within a two-dimensional region $U$ of interest and with $t$ referring to time. Fluid trajectories generated by $v(x, t)$ are denoted $x(t; t_0, x_0)$, with $x_0$ referring to the initial position of the trajectory at time $t_0$. These trajectories solve the differential equation

$$\dot{x} = v(x, t) \tag{1.1}$$

and generate the flow map

$$F_{t_0}^{t}(x_0) := x(t; t_0, x_0), \tag{1.2}$$

which takes an initial position $x_0$ at time $t_0$ to its current position at time $t$.

The right Cauchy–Green strain tensor field associated with the flow map is defined as $C_{t_0}^{t}(x_0) = \nabla F_{t_0}^{t}(x_0)^{\top} \nabla F_{t_0}^{t}(x_0)$, with eigenvalues $\lambda_i(x_0)$ and eigenvectors $\xi_i(x_0)$ satisfying

$$C_{t_0}^{t}\xi_i = \lambda_i \xi_i, \quad |\xi_i| = 1, \, i = 1, 2; \, 0 < \lambda_1 \leqslant \lambda_2, \, \xi_1 \perp \xi_2. \tag{1.3}$$

In Haller & Beron-Vera (2013) we sought the time $t_0$ positions of Lagrangian vortex boundaries as closed stationary curves of the averaged Lagrangian strain. Such curves

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turned out to coincide with the zero-energy solutions of a one-parameter family of variational problems defined as

$$\delta\mathcal{E}_\lambda(\gamma)=0,\quad \mathcal{E}_\lambda(\gamma)=\oint_\gamma \langle r'(s), \boldsymbol{E}_\lambda(r(s))r'(s)\rangle\,\mathrm{d}s,\ \ \lambda\in\mathbb{R}^+. \tag{1.4}$$

Here, the strain energy functional $\mathcal{E}_\lambda(\gamma)$ is defined through the generalized Green–Lagrange strain tensor family

$$\boldsymbol{E}_\lambda(x_0)=\tfrac{1}{2}[\boldsymbol{C}_{t_0}^t(x_0)-\lambda^2\boldsymbol{I}],\quad \text{where } \boldsymbol{I} \text{ is the identity matrix.} \tag{1.5}$$

Consider the flow domain

$$U_\lambda=\{x_0\in U \mid \lambda_1(x_0)<\lambda^2<\lambda_2(x_0)\}, \tag{1.6}$$

where the tensor field $\boldsymbol{E}_\lambda$ has two non-zero eigenvalues of opposite sign. Then the quadratic function

$$g_\lambda(u,u)=\langle u, \boldsymbol{E}_\lambda u\rangle \tag{1.7}$$

defines a Lorentzian metric (Beem, Ehrlich & Kevin 1996) on $U_\lambda$, with signature $(-,+)$ inherited from the eigenvalue configuration of $\boldsymbol{E}_\lambda$. The zero-energy solutions of (1.4) are therefore precisely the closed null geodesics of the Lorentzian metric $g_\lambda$, which satisfy one of the two differential equations

$$r'(s)=\eta_\lambda^\pm(r(s)),\quad \eta_\lambda^\pm(r)=\sqrt{\frac{\lambda_2(r)-\lambda^2}{\lambda_2(r)-\lambda_1(r)}}\,\xi_1(r)\pm\sqrt{\frac{\lambda^2-\lambda_1(r)}{\lambda_2(r)-\lambda_1(r)}}\,\xi_2(r). \tag{1.8$a$,$b$}$$

In Haller & Beron-Vera (2013) we concluded that closed orbits of (1.8) (termed closed *$\lambda$ lines*) must necessarily encircle metric singularities of $g_\lambda$. Such singularities occur at points $x_0$ where $\lambda_1(r)=\lambda_2(r)$ holds for the eigenvalues of the Cauchy–Green strain tensor.

The Lorentzian metric interpretation discussed above implies a geometric analogy between coherent Lagrangian vortex boundaries and photon spheres in cosmology. Below, we give more detail on this analogy, followed by an improved version of its summary with a more relevant reference.

## 2. More on the analogy with photon spheres

In the vicinity of any $\lambda$ line in $U_\lambda$, the vector fields $\xi_i$ define a curvilinear coordinate system with pointwise orthogonal coordinate lines. Direct substitution of $\xi_i$ into the metric (1.7) gives

$$g_\lambda(\xi_i,\xi_i)=\tfrac{1}{2}(\lambda_i-\lambda^2), \tag{2.1}$$

showing that $g_\lambda(\xi_1,\xi_1)<0$ and $g_\lambda(\xi_2,\xi_2)>0$ everywhere in $U_\lambda$. This shows that the $\xi_1$ trajectories form the time-like coordinates and the $\xi_2$ trajectories form the space-like coordinates of the metric $g_\lambda$ in $U_\lambda$ (Beem *et al.* 1996). We further note that $g_\lambda(\eta_\lambda^\pm,\eta_\lambda^\pm)=0$, and hence a $\lambda$ line is nowhere space-like in the language of Lorentzian geometry. Given that our closed null geodesics are nowhere space-like hypersurfaces built out of null geodesics, they are photon surfaces by the general definition of Claudel, Virbhadra & Ellis (2001).

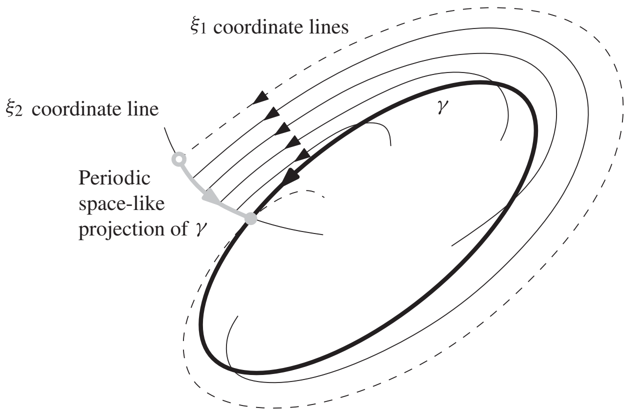


FIGURE 1. A closed null geodesic $\gamma$ of the Lorentizan metric $g_\gamma$ has a periodic space-like projection along the time-like coordinate lines. This space-like projection has a jump discontinuity due to the low dimensionality of the space–time, but evolves periodically along with the underlying closed null geodesic. This is in contrast to general $\lambda$ lines which have aperiodic space-like projections.

Next, we note that

$$\left\langle \eta_\lambda^\pm(r), \xi_1(r) \right\rangle = \sqrt{\frac{\lambda_2(r)-\lambda^2}{\lambda_2(r)-\lambda_1(r)}} \in (0, 1), \qquad r \in U_\lambda. \tag{2.2}$$

Consequently, trajectories of the $\xi_1(r)$ line field ($\xi_1$ coordinate lines) intersect any closed $\lambda$ line $\gamma$ transversely, with an angle of intersection that is always less than $\pi/2$, as sketched qualitatively in figure 1. In the same figure, we also show a representative trajectory of the $\xi_2(r)$ line field (a $\xi_2$ coordinate line), which is pointwise orthogonal to the $\xi_1$ coordinate lines by construction. We conclude that the projection of $\gamma$ onto a space-like submanifold along the time-like coordinates results in a periodic (albeit discontinuous) space-like orbit (cf. figure 1).

In summary, a closed orbit $\gamma$ of the $\eta_\lambda^\pm(r)$ vector field is a photon surface of the $(U_\lambda, g_\lambda)$ space–time. Geodesics forming this photon surface have periodically moving projections on the space-like coordinates, with the projection taken along the time-like coordinate lines.

The orbit $\gamma$, therefore, satisfies a plausible extension of the Claudel–Virbhadra–Ellis definition of a photon sphere (Claudel *et al.* 2001) from symmetric higher-dimensional space–times to non-symmetric two-dimensional space–times. This extension relaxes the requirement for a rotational symmetry and smoothness of the projected spatially periodic orbits. Both of these features are unattainable in curved two-dimensional space–times, and hence can reasonably be waived.

## 3. Revised wording of the cosmological analogy

In view of the above discussion, the wording of the mathematical analogy between coherent Lagrangian eddy boundaries and photon spheres in cosmology (Haller & Beron-Vera 2013, p. 731, para. 2) should be revised as follows.

> The closed $\lambda$ lines we have been seeking are therefore closed null geodesics of $g_\lambda$. Trajectories spanning these geodesics project along the

local time-like coordinates (trajectories of the $\xi_1$ vector field) onto periodic trajectories on the local space-like coordinates (trajectories of the $\xi_2$ vector field). In cosmology, the projection of some families of null geodesics from space–time onto the space-like variables also produces closed trajectories. Such families of null geodesics are often referred to as photon spheres, as their spatial projections trap photons orbiting around black holes (Claudel *et al.* 2001). In the cosmological context, null geodesics are tangent to light cones, which in our case are formed by the two vectors $\eta_\lambda^\pm(x_0)$ at $x_0$ (figure 2 of Haller & Beron-Vera 2013).

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